Supporting Material: Near-Field Radiative Heat Transfer in Graphene Plasmonic Nanodisk Dimers

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FIG. S1. Schematic representation of a graphene nanostructure and the coordinates considered for the electrostatic theory.

S1. ELECTROSTATIC THEORY FOR GRAPHENE NANOSTRUCTURES

A. General formulation of electrostatic coupling between 2D nanostructures

Consider a graphene nanostructure as conducting surface with no thickness (???). In the electrostatic regime, the electromagnetic response of a graphene nanostructure located between two media with dielectric constants \( \varepsilon_1 \) and \( \varepsilon_2 \) is given by the electric potential \( \psi \) that results from the induced surface charge \( \rho \) as:

\[
\psi_{\text{out}}(\gamma) = \frac{1}{\varepsilon_0\varepsilon_m} \int_S d^2\gamma' \ K(\gamma, \gamma') \rho(\gamma').
\] (S1)

Here, \( K(\gamma, \gamma') = \frac{1}{4\pi|\gamma - \gamma'|} \) is the Green’s function anywhere outside the graphene nanostructure and \( \gamma = \gamma_\parallel + \gamma_\perp \) is a vector defined by components on the surface, \( \gamma_\parallel \) and normal to the surface, \( \gamma_\perp \), as shown in ???. The integration domain, \( S \), is defined along the surface of the graphene nanostructure.

The induced surface charge is related to the induced electric potential at the surface, \( \psi_{\text{in}} \), by the continuity equation \( (\nabla_\parallel \cdot J = i\omega\rho \), where \( \nabla_\parallel \) is the two dimensional nabla operator) and by representing the surface current in terms of the electric potential as \( J = -f(\gamma_\parallel)\sigma(\omega)\nabla_\parallel \psi_{\text{in}} \), which gives the expression:

\[
i\omega\rho = -\nabla_\parallel \cdot \left[ \sigma(\omega)f(\gamma_\parallel)\nabla_\parallel \psi_{\text{in}} \right].
\] (S2)

Here, \( \sigma(\omega) \) is the optical conductivity of graphene taken from Eq. (1) from the main text, and \( f(\gamma_\parallel) \) is a filling function that is unity inside the graphene nanostructure domain and zero elsewhere.
We express $\psi_{in}$ as a function of $\rho$ using an appropriate choice of Green’s function as:

$$\psi_{in}(\gamma_\parallel) = \frac{i\omega}{\sigma(\omega)} \int_S d^2\gamma_\parallel' \, g(\gamma_\parallel, \gamma_\parallel') \rho(\gamma_\parallel'), \quad (S3)$$

where $g(\gamma_\parallel, \gamma_\parallel')$ is a Green’s function satisfying the relations

$$f \nabla_\parallel^2 g = -\delta(\gamma_\parallel - \gamma_\parallel')$$

$$[\hat{u} \cdot \nabla_\parallel g]_{\gamma_\parallel = \xi_0} = 0,$$

where $\xi_0$ represents the contour of the nanostructure and $\hat{u}$ is a vector normal to the edge of the nanostructure.

In the definition of $\psi_{in}$ we assume no tunneling of charges at nanostructure’s edge, i.e.

$$[\hat{u} \cdot \nabla_\parallel \psi_{in}]_{\gamma_\parallel = \xi_0} = 0,$$

The electrostatic interaction of a system with $N$ graphene nanostructures and an external field with an electric potential, $\psi_{ext}$, is modeled by using $\psi_{in}$ and the boundary condition at the surface of each disk. For example, the continuity of the electric potential at the surface of a nanostructure $n$ gives

$$\psi_{in}^n(\gamma_\parallel' - \gamma_n) = \sum_{\nu=1}^N \psi_{out}^\nu(\gamma_\parallel' - \gamma_\nu) + 2\psi_{ext}(\gamma_\parallel'). \quad (S4)$$

The summation on the right-hand side of $\psi_{in}$ accounts for the interaction of the nanostructure $n$ with the rest of the nanostructures $\nu$ ($\nu = 1, 2, ..., N$), including the interactions with its own induced electrostatic potential, i.e $\psi_{out}^n(\gamma_\parallel' - \gamma_n)$. A factor of two is considered in the external field potential as the field acts on both sides of the graphene surface.

We construct a system of $N$ equations by applying $\psi_{in}$ to each nanostructure. The solution is found by expanding the surface charge distribution $\rho_\nu, \nu = 1, 2, ..., N$ into a set of normal modes and taking the inner product on both sides of $\psi_{in}$. The surface current expansion is represented as

$$\rho_\nu(\gamma_\parallel) = \sum_{j}^\infty c_j^\nu \beta_j^\nu(\gamma_\parallel), \quad (S5)$$

where $c_j^\nu$ and $\beta_j^\nu$ are the $j$-mode constant and function basis in the nanostructure $\nu$.

We finally derive the system of equations

$$\frac{1}{2} \left[ \sum_{\nu=1}^N G_\nu - G^0 \right] \mathbf{c} = \mathbf{X}, \quad (S6)$$
where the interaction matrices $G^\nu$ and $G^0$ are composed of $N \times N$ sub-matrices whose elements are given by:

$$G^\nu_{m,n} = \begin{cases} \frac{i\omega}{\sigma(\omega)} \langle \beta^m_{\nu}, g^\nu \ast \beta^n_{\nu} \rangle & \text{if } m = n = \nu \\ 0 & \text{Otherwise} \end{cases} \quad (S7)$$

and $\mathbf{X}$ is a column matrix composed of $N$ sub-matrices of the form $X^m_{\nu} = \langle \beta^m_{\nu}, \psi_{\text{ext}} \rangle$. $\langle \cdot, \cdot \rangle$ represents the standard inner product $\langle u, v \rangle = \int d^2\gamma (u^* v)$, and the $*$ symbol represents the transformation

$$Q \ast \beta = \int_{S^r} d^2\gamma' \frac{Q(\gamma, \gamma')}{\beta(\gamma')}.$$

The size of each matrix is given by the number of modes in the expansion, which should be large enough to reach convergence. The system in $\nu$, together with $\nu$, gives the magnitude of the expansion coefficients $c$ and thus the total response of the system of $N$ nanostructures.

**B. Analytical solution for graphene nanodisks**

For a disk of radius $R$, a convenient expansion of the surface charge is given by an orthogonal basis in cylindrical coordinates $(r, \phi, z)$ of the form

$$\beta_{kl}(r, \phi) = P_k^{(L,0)}(1 - 2x^2)x^L \Theta(1 - x) \frac{1}{\sqrt{2\pi}} e^{i\phi l}, \quad (S9)$$

where $L = |l|$, $x = r/R$, $P_k^{(L,0)}$ are Jacobi polynomials, $\Theta$ is the step function, and $k$ and $l$ represent the radial and angular modes. The advantage of using this basis lies in the identity

$$\int_0^R \beta_{kl}(r, \phi) J_l(pr)r dr = R^{-1} J_{L+2k+1}(R) \frac{1}{\sqrt{2\pi}} e^{i\phi l}. \quad (S10)$$

The electrostatic Green’s function $K(\gamma, \gamma') = \frac{1}{4\pi |\gamma - \gamma'|}$ in cylindrical coordinates is given by

$$K(\gamma, \gamma') = i \frac{1}{2\pi} \sum_{l=-\infty}^{+\infty} e^{i(\phi - \phi')} \int_0^\infty Z_l(ik_z r) J_l(ik_z r') \cos[k_z(z - z')] dk_z, \quad (S11)$$

where $k_z$ is the wavevector in the direction normal to the plane of the disk and

$$Z_l(ik_z r) = \begin{cases} H_l(ik_z r) , & r > R \\ J_l(ik_z r) , & r \leq R \end{cases}.$$
FIG. S2. Coordinates that describe the interaction between two disks labeled \( m \) and \( n \). (a) Top view, (b) Lateral view.

where \( H_l \) and \( J_l \) are the Hankel and Bessel functions. Similarly, \( g^\nu(\gamma_\parallel,\gamma'_\parallel) \) can be expressed as:

\[
g^\nu(\gamma_\parallel,\gamma'_\parallel) = \frac{1}{2\pi} \sum_{l=\infty}^{\infty} e^{i(l\phi-\phi')} \left[ \frac{(r'r')^L}{2L} + \int_0^\infty J_l(ik_zr)J_l(ik_zr')k_z^{-1}dk_z \right]. \tag{S12}
\]

Using ?? and the identity from ??, together with the corresponding Green’s functions from ????, we calculate the self-interaction matrices elements from ????. The result is?

\[
G^{\nu,\nu}_{l_1k_1,l_2k_2} = \frac{i\omega R^4}{\sigma(\omega)} \cdot \begin{cases} 
[8L(L+1)]^{-1}\delta_{l_1,l_2} & k_1 = k_2 = 0 \\
[4(L+2k_1)(L+2k_1+1)(L+2k_1+2)]^{-1}\delta_{l_1,l_2} & k_1 = k_2 \neq 0 \\
[8(L+2k_1+1)(L+2k_1+2)(L+k_1+3)]^{-1}\delta_{l_1,l_2} & k_1 = k_2 + 1 \\
[8(L+2k_1+1)(L+2k_1+2)(L+k_1+3)]^{-1}\delta_{l_1,l_2} & k_1 + 1 = k_2 \\
0 & \text{otherwise}
\end{cases}
\]

\[
G^{0,\nu}_{l_1k_1,l_2k_2} = \frac{R^3}{2\varepsilon_0\varepsilon_h} \cdot \frac{(-1)^{k_1-k_2+1}}{\pi[4(k_1-k_2)^2-1](L+k_1+1k_2+1/2)(L+k_1+k_2+3/2)} \delta_{l_1,l_2},
\]

where \( L = |l_1| = |l_2| \), and \( \delta_{l_1,l_2} \) is the Kronecker delta that takes the value 1 when \( l_1 = l_2 \), and 0 otherwise. The presence of \( \delta_{l_1,l_2} \) in the previous equations means that the interaction between different angular modes is forbidden.

The interaction matrix elements between two different disks \( m \) and \( n \), \( G^{0,\nu}_{l_1k_1,l_2k_2} \), are

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obtained by using a transformation of the Green’s function from Cartesian to cylindrical coordinates. This procedure allows for a simplification of the calculation by directly using ??.

We consider the interaction of two disks \( m \) and \( n \), as shown in ??, Using Green’s functions in the plane wave representation the interaction elements are expressed as:

\[
G_{lm,k_m,l_n,k_n}^{0,mn} = \frac{i}{4\pi^2 \varepsilon_0 \varepsilon h} \int_{S_m} d^2 \gamma_n \int_{S_n} d^2 \gamma_m \int_{-\infty}^{\infty} d^2 k_l \beta_{lm,k_m} \ast (\gamma_n^m) e^{ik_l (\gamma_n^m - \gamma_n^n)} \frac{e^{ik_{oz} \Delta z}}{2k_{oz}} \beta_{lm,k_n} (\gamma_n^n),
\]

where \( \Delta z = |z_m - z_n| \), \( k_l = k_x \hat{x} + k_y \hat{y} \), and \( k_{oz} = i \sqrt{k_x^2 + k_y^2} \).

Using the change of variables \( k_r = \sqrt{k_x^2 + k_y^2} \) and \( \varphi = \arctan(k_y/k_x) \), we obtain a new expression for ?? where the surface integrals are decoupled as

\[
G_{lm,k_m,l_n,k_n}^{0,mn} = \frac{1}{4\pi^2 \varepsilon_0 \varepsilon h} \int_0^{2\pi} d\varphi \int_0^{\infty} k_r dk_r e^{-k_r \Delta z} \frac{e^{ik_r d \cos(\alpha - \varphi)}}{2k_r} \times \int_{S_m} d^2 \gamma_n \beta_{lm,k_m} \ast (\gamma_n^m) e^{ik_{rm} \cos(\phi_m - \varphi)} \times \int_{S_n} d^2 \gamma_n \beta_{lm,k_n} (\gamma_n^n) e^{ik_{rn} \cos(\phi_n - \varphi + \pi)},
\]

where we use the identity \( e^{ik \parallel \gamma} = e^{ik_r \cos(\varphi - \phi)} \), where \( \phi \) is the angle formed by \( \gamma \) and the \( x \) axis, \( \alpha \) is the angle formed between \( d \) and the \( x \) axis (??).

The surface integrals from ?? can be simplified using the identity ??

\[
e^{ik_r \cos \alpha} = \sum_{q=-\infty}^{+\infty} i^q J_q(k_r \varphi) e^{iq \alpha},
\]

which gives

\[
\int_{S_m} d^2 \gamma_n \beta_{lm,k_m} \ast (\gamma_n^m) e^{ik_{rm} \cos(\phi_m - \varphi)} = R_m^2 \sqrt{2\pi} i^m B_{lm,k_m}^m (R_m k_r) e^{-ilm \varphi}
\]

\[
\int_{S_n} d^2 \gamma_n \beta_{lm,k_n} (\gamma_n^n) e^{ik_{rn} \cos(\phi_n - \varphi + \pi)} = R_n^2 \sqrt{2\pi} i^{-l-n} B_{ln,k_n}^n (R_n k_r) e^{i(l-n) \varphi},
\]

where we used ?? to get \( B_{kl}^n(p) = \left( \frac{L}{T} \right)^l p^{-1} J_{L+2k+1}(p) \).

Inserting ??? into ?? gives

\[
G_{lm,k_m,l_n,k_n}^{0,mn} = \frac{R_m^2 R_n^2}{\varepsilon_0 \varepsilon h} \int_0^{\infty} dk_r B_{lm,k_m}^m (R_m k_r) \times \frac{e^{-k_r \Delta z}}{2} \left[ \frac{1}{2\pi} \int_0^{2\pi} e^{ik_r d \cos(\alpha - \varphi)} i^{(l_m - l_n)} e^{i(l_m - l_n) \varphi} d\varphi \right] \times B_{ln,k_n}^n (R_n k_r).
\]

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Using ?? the term in the square brackets is reduced and the following expression is obtained

\[ G_{0,0}^{m,m} = \frac{R_n^2}{2\varepsilon_0 \varepsilon_h} \int_0^\infty dk_r B_{k_m,l_m}^m(R_m k_r) J_{l_m-l_m}(k_r d) e^{i\alpha(l_m-l_m)} e^{-k_r \Delta z} B_{k_n,l_n}^n(R_n k_r), \]  

(S17)

which is valid when \( d < R \). In the case where \( d > R \), the Bessel function, \( J_{l_m-l_m}(k_r d) \), needs to be replaced by the Hankel function, \( H_{l_m-l_m}(k_r d) \), as only outgoing waves exist in this domain. This last change, however, leads to numerical instability in the integration, which can be solved by the change of variables \( k_r = ik_z \). Thus, an appropriate expression for the domain \( d > R \) is given by

\[ G_{0,0}^{m,m} = \frac{i R_n^2 R_m^2}{2\varepsilon_0 \varepsilon_h} \int_0^\infty dk_z B_{k_m,l_m}^m(i R_m k_z) H_{l_m-l_m}(ik_z d) e^{i\alpha(l_m-l_m)} \cos(k_z \Delta z) B_{k_n,l_n}^n(i R_n k_z). \]  

(S18)

The terms \( J_{l_m-l_m}(k_r d) e^{i\alpha(l_m-l_m)} \) in ?? and \( H_{l_m-l_m}(ik_z d) e^{i\alpha(l_m-l_m)} \) in ?? represent the strength of interaction between different angular modes. At \( d = 0 \), this terms becomes \( \delta_{l_m,l_n} \), meaning that in the co-axial disks configuration the interaction between different angular modes is forbidden. In our study, co-planar disks lie on the \( x \) axis, i.e., \( d = \Delta x \) and \( \alpha = 0 \).

The number of modes was determined by analyzing the convergence in the near-field thermal conductance, \( G_{rad}(\omega) \) (see ??). At separations, \( \Delta x \) or \( \Delta z \), larger than 0.01\( D \), convergence with an error of less than \( \pm 1\% \) was reached for \( l = (0, \pm 1, \pm 2, \ldots \pm 7) \) and \( k = (0, 1, \ldots 8) \). Smaller separations require larger number of modes for convergence.

S2. THEOREY OF RADIATIVE HEAT TRANSFER

A. Radiation flux spectrum in the electrostatic limit

Consider two graphene disks, labeled 1 and 2. The electromagnetic energy dissipation in disk 2 due to the electrostatic fields from a random distribution of fluctuating charges in disk 1 is given by

\[ P_{1 \to 2}(\omega) = \left\langle \frac{\omega}{2} \text{Im} \left( \rho^2, \Psi_{out}^{12} \right) \right\rangle, \]  

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where $\langle \cdots \rangle$ denotes the ensemble average. Since the electric potential at the surface of each disk is continuous ($i.e., \Psi_{\text{out}}^{12} = \Psi_{\text{in}}^{22}$) we can rewrite the previous expression as

$$P_{1 \rightarrow 2}(\omega) = \left\langle \frac{\omega}{2} \text{Im} \left( \rho^2, \Psi_{\text{in}}^{22} \right) \right\rangle = \left\langle \frac{\omega}{2} \text{Im} \left( \rho^2, \frac{i\omega}{\sigma(\omega)} (g^2 \rho^1) \right) \right\rangle$$

$$= \left\langle \frac{\omega}{2} \text{Im} [c^* G^2 c] \right\rangle = \frac{\omega}{2} \left\langle c^* G_{\text{AH}}^2 c \right\rangle$$

$$= -i \frac{\omega}{2} \text{Tr} \left[ (XX^*) W^* G_{\text{AH}}^2 W \right]. \quad (S19)$$

The external source is driven by the fluctuating charges in disk 1, $\rho^1_{\text{fl}}$. Thus,

$$\langle XX^* \rangle_{\text{in},mn} = \begin{cases} \langle x^1 x^1^* \rangle_{\text{in},ij} & \text{if } m = n = 1 \\ 0 & \text{Otherwise}, \end{cases}$$

where $x^1$ is a column vector whose elements are given by $x^1_i = \frac{i\omega}{\sigma(\omega)} \langle \beta^1_i, g^1 \rho^1_{\text{fl}} \rangle$. The elements of the ensemble-average $\langle x^1 x^1^* \rangle$ can be expressed as

$$\langle x^1 x^1^* \rangle_{ij} = \left\langle \frac{\beta^1_i}{\sigma(\omega)} \left( g^1 \rho^1_{\text{fl}} \right) \left( \rho^1_{\text{fl}}^* \right) \right\rangle$$

$$= \left\langle \frac{\beta^1_i}{\sigma(\omega)} \left( g^1 \rho^1_{\text{fl}} \right) \left( \rho^1_{\text{fl}}^* \right) \right\rangle$$

$$= \left\langle \frac{\beta^1_i}{\sigma(\omega)} \left( g^1 \rho^1_{\text{fl}} \right) \left( \rho^1_{\text{fl}}^* \right) \right\rangle$$

$$= \left\langle \frac{\beta^1_i}{\sigma(\omega)} \left( g^1 \rho^1_{\text{fl}} \right) \left( \rho^1_{\text{fl}}^* \right) \right\rangle$$

$$= \left\langle \frac{\beta^1_i}{\sigma(\omega)} \left( g^1 \rho^1_{\text{fl}} \right) \left( \rho^1_{\text{fl}}^* \right) \right\rangle$$

where $\bar{\sigma}(\omega)$ represent the complex conjugate of the optical conductivity of graphene.

In the last step of ?? we consider the fluctuation-dissipation theorem:

$$\left\langle \rho^1_{\text{fl}}(\gamma_{||}) \rho^1_{\text{fl}}^*(\gamma_{||}^\prime) \right\rangle = \frac{1}{\omega \pi} \text{Im} \left[ \chi \left( \omega, \gamma_{||}, \gamma_{||}^\prime \right) \right] f_{\text{BE}}(\omega, T_1).$$

Here, $\chi$ is the linear susceptibility of the surface charge in response to an electric potential.

According to linear response theory, the relation between the linear susceptibility and the surface charge density is given by

$$\rho^1(\gamma_{||}) = \int_{S^1} \chi \left( \omega, \gamma_{||}, \gamma_{||}^\prime \right) \Psi_{\text{ext}}(\gamma_{||}^\prime) d^2 \gamma_{||}^\prime.$$

Thus, using Eqs. (12) and (14) of the main text gives the following representation of $\chi$ in terms of the mode expansion

$$\chi \left( \omega, \gamma_{||}, \gamma_{||}^\prime \right) = \sum_{ij} \beta^1_i(\gamma_{||}) \left[ \hat{g}^{-1}\right]_{ij} \beta^1_j(\gamma_{||}^\prime). \quad (S21)$$
where $\hat{g}_{ij}^1 = \frac{i\omega}{\sigma(\omega)} \langle \beta_i^1, g^1 \beta_j^1 \rangle$.

Substituting ?? into ?? gives

$$\langle XX^* \rangle = \frac{i}{\pi \omega} G_{AH}^1 f_{BE}(\omega, T_1).$$

(S22)

Finally, by inserting ?? into ?? we derive the following

$$P_{1\to 2}(\omega) = \Phi_{12}(\omega)f_{BE}(\omega, T_1),$$

where $\Phi_{12}(\omega)$ is the flux spectrum defined as

$$\Phi_{12}(\omega) = \frac{1}{2\pi} \text{Tr} [G_{AH}^1 W^* G_{AH}^2 W].$$

(S23)

S9. NON-DIMENSIONAL NEAR-FIELD THERMAL CONDUCTANCE

For the case of two identical graphene disks of radius $R$, ?? can be converted to a non-dimensional form as??

$$\left[ \tilde{G}^1 + \tilde{G}^2 - \eta(\omega) \tilde{G}^0 \right] c = \tilde{X},$$

(S24)

where

$$\tilde{G}_{\nu, mn}^{\nu, mn} = \frac{1}{R^2} \begin{cases} \langle \beta_m^\nu, g^\nu \beta_n^\nu \rangle & \text{if } m = n = \nu \\ 0 & \text{Otherwise} \end{cases}$$

$$\tilde{G}_{\nu, mn}^{0, mn} = \frac{1}{R^2} \langle \beta_m^\nu, K \beta_n^\nu \rangle$$

$$\tilde{X}_m^\nu = \frac{\sigma(\omega)}{i\omega} \langle \beta_m^\nu, \psi_{ext} \rangle.$$

The term $\eta$ is defined as

$$\eta(\omega) = \frac{\sigma(\omega)}{i\omega} \frac{1}{R}$$

and $\tilde{G}_{\nu, mn}^{0, mn}$ is explicitly defined as:

$$\tilde{G}_{\nu, mn}^{0, mn} = \frac{1}{2} \int_0^\infty dp \begin{cases} B_{kn, l_m}(p) J_{n-l_m} \left( 2p \frac{\Delta}{D} \right) e^{i\alpha(l_n-l_m)} e^{-2p \Delta^2} B_{kn, l_m}^*(p) & \text{if } d/D < 1/2 \\ B_{kn, l_m}(ip) H_{n-l_m} \left( 2p \frac{\Delta}{D} \right) e^{i\alpha(l_n-l_m)} \cos \left( 2p \frac{\Delta}{D} \right) B_{kn, l_m}^*(ip) & \text{if } d/D > 1/2, \end{cases}$$

where $p = Rk_\nu$. 

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FIG. S3. Non-dimensional thermal conductance $G_{NF}/k_B\omega_R$ for (a) co-axial and (b) co-planar disks configurations. $h\omega_R/k_B T = 10$ and the disk diameter ranges from 100 to 500 nm. All calculations consider the full form of the optical conductivity [Eq. (1) of the main text]. The red lines indicate the range of the results compared to $D = 100$ nm.

The electrostatic problem in the form of ?? is uniquely determined by $\eta$, $\Delta z/D$ and $d/D$. Furthermore, at frequencies satisfying $h\omega/E_f < 1$ interband transitions dominate, and for low temperatures ($k_B T/E_f \ll 1$) the optical conductivity reduces to

$$\sigma(\omega) = \frac{e^2}{\pi\hbar^2} \frac{i}{\omega + i\tau - 1}. \tag{S25}$$

Thus, for $(\omega_R\tau)^{-1} \ll 1$, $\eta$ can be simplified to

$$\eta = \frac{\omega^2}{\omega^2}, \tag{S26}$$

where

$$\omega_R = \sqrt{\frac{2E_f e^2}{\hbar^2 \varepsilon_0 \varepsilon_h D}}$$

is the characteristic plasmon frequency.

The flux spectrum from ?? can now be written as a function of the parameters $\omega/\omega_R$, $\Delta z/D$ and $d/D$. Similarly, the near-field radiative thermal conductance given by Eq. (2) of the main text in its non-dimensional form is given by

$$\frac{G_{NF}}{k_B\omega_R} = \int_0^\infty \Phi_{12} \left( \frac{\omega}{\omega_R}, \frac{\Delta z}{D}, \frac{\Delta x}{D} \right) \left( \frac{\omega}{\omega_R} \right) \frac{d}{d(k_B T/h\omega_R)} \Phi_{BE} \left( \frac{\omega}{\omega_R}, \frac{k_B T}{\hbar \omega_R} \right) d \left( \frac{\omega}{\omega_R} \right), \tag{S27}$$

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FIG. S4. Non-dimensional thermal conductance $G_{NF}/k_B\omega_R$ between two co-planar disks as a function of $\Delta x/D$. The transition from the weak coupling regime occurs at $(\Delta x/D)_{t-w} \approx 3.2$.

$i.e.$ the non-dimensional thermal conductance $G_{NF}/k_B\omega_R$ is uniquely determined by $\omega/\omega_R$, $\Delta z/D$, $\Delta x/D$ and $\hbar\omega_R/k_B T$.

$G_{NF}/k_B\omega_R$ is plotted as function of $\Delta z/D$ and $\Delta x/D$ for $\hbar\omega_R/k_B T = 10$ and disk diameters ranging from 100 to 500 nm in ??(a) and ??(b). The results were calculated using ?? with $\tau = 0.6$ fs and temperatures adjusted such that $\hbar\omega_R/k_B T = 10$. The disagreement in the curves comes from the influence of the parameter $\omega_R\tau$. Taking the results of $D = 100$ nm as a reference $[(\omega_R\tau)^{-1} = 4.2 \times 10^{-3}]$, the maximum range of the curves is 15% to 5% for co-axial disks and from 9% to 5% for co-planar disks.

?? is used to determine the length scales $\Delta z/D$ and $\Delta x/D$ that determine the domains of the strong coupling, transition and weak coupling regimes. The results for co-axial disks are plotted in Fig. 4 of the main text. The results for co-planar disks are plotted in ???. Only the weak coupling regime can be distinguished which is characterized by the $(\Delta x)^{-6}$ scaling. Considering a 0.2% deviation from the $(\Delta x)^{-6}$ scaling, it is determined that the length scale that defines the change in the curve from weak coupling to transition regime is $\Delta x/D \approx 3.2$.

S4. MISMATCHED DIAMETERS (CO-PLANAR DISKS)
FIG. S5. Effect of breaking symmetry in the disk diameter on the thermal conductance for co-planar disk configuration. (a) Relative thermal conductance of a heterodimer with respect to that of a homodimer. $D_1 = 100$ and $D_2$ varies from 50 to 150 nm. (b) Spectral thermal conductance of the heterodimer for selected cases

In (a), $G_{NF}/G_{D2/D1=1}^{D1}$ is plotted for co-planar disks, with $\Delta x_e/D_1 = 0.05$, 0.1, 0.3, 0.5, and 3, $D_1 = 100$ nm, and 50 nm $< D_2 < 150$ nm. Here, $\Delta x_e$ is the distance between the disks edge. This allows a better comparison between different $D_2/D_1$ ratios, given that near-field coupling between co-planar disks is mainly between the disk edges. For the majority of cases, the heterodimers show a reduced thermal conductance compared to the homodimer. For $\Delta x/D_1 = 0.05$, an enhancement up to $G_{NF}/G_{D2/D1=1}^{D1} = 1.01$ is observed in the range $1.0 < D_2/D_1 < 1.04$. In (b) the spectral thermal conductance is plotted for $D_2/D_1 = 0.8$, 1.0, and 1.2, and $\Delta z/D_1 = 0.05$, 0.3, and 3. Similar to the results from Fig. 6(b) from the main text, more modes interact when the distance between disk edges is reduced, adding more channels that enhance the near-field radiation heat transfer between the disks.
F. J. García de Abajo, ACS Photonics 1, 135 (2014).


